Carrollian fields at time-like infinity

Jakob Salzer

Université libre de Bruxelles

Carroll Workshop, Erwin Schrödinger Institut Wien
April 2nd, 2024

UNIVERSITÉ LIBRE DE BRUXELLES

fnrs LA LIBERTÉ DE CHERCHER

based on 2402.05190 w. E. Have, K. Nguyen, S. Prohazka

Motivation

- Carrollian structures appear in gravity in various places.
- Most prominently in recent times: conformal Carroll structure at null infinity of asymptotically flat spacetime.

Major Research Question

Is there a holographic description of AF spacetimes as a Carrollian theory?

- Basic observation: Asymptotically, massless fields can be interpreted as conformal carrollian fields $\Leftrightarrow$ massless scattering amplitudes can be written as conformal carrollian correlators
- A full-fledged holographic dual should be able to describe massive particles as well.
- Natural question: are massive particles also amenable to a Carrollian description?

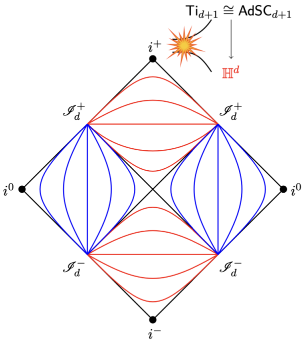
This talk

Question

Is there an asymptotic Carrollian picture for massive particles?

Short Answer

At late times, massive fields can be described as Carrollian fields, subject to Carrollian field equations, on a curved Carrollian manifold called Ti/AdS-Carroll. Note that this space is of the same dimension as the bulk.



There are more things (Carrollian) in Heaven and Earth (?), Horatio, than are dreamt of in your philosophy...

Outline

Outline

Defining Ti/AdS-Carroll

- Carroll structure (n, q) of four-dimensional Ti:

$$n = \partial_\tau\,, \quad q = 0\, d\tau^2 + h_{ab}\, dx^a dx^b\,, \quad h_{ab}\, dx^a\, dx^b = dr^2 + \sinh^2 r \;\; d\Omega_2.$$

  with h the metric on $EAdS_3/\mathbb{H}_3$.

- Symmetries of the Carroll structure $\mathcal{L}_\xi q = \mathcal{L}_\xi n = 0$:

$$\xi = \underbrace{S(x^a)}_{\text{Ti-supertranslations}} \partial_\tau + \underbrace{\xi^a}_{\text{KV's of } \mathbb{H}_3\,:\, \mathfrak{so}(3,1)} \partial_a$$

- The Poincaré subgroup is given by

$$\xi_{J_{ab}} = x_a \partial_b - x_b \partial_a$$

$$\xi_{B_a} = -r \coth r \left(\partial_a - \frac{x_a}{r}\partial_r\right) - \frac{x_a}{r}\partial_r \qquad \text{Carroll translations}$$

$$\xi_H = -\cosh r\, \partial_\tau$$

$$\xi_{P_a} = \sinh r\, \frac{x_a}{r} \partial_\tau \qquad \text{Carroll boosts}$$

Ti/AdS-Carroll as a homogeneous space

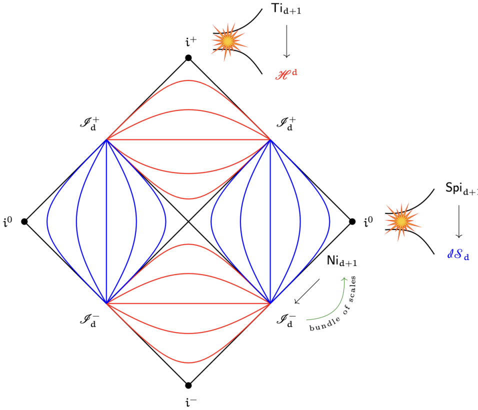
- Ti is a homogeneous space of Poincaré

$$\mathrm{Ti}_4 \cong \frac{\mathrm{ISO}(3,1)}{\mathrm{ISO}(3)}$$

- Compare with Minkowski and flat Carrollian space ($n = \partial_\tau, q = \delta_{ab}\, dx^a\, dx^b$)

$$\mathbb{M}_4 \cong \frac{\mathrm{ISO}(3,1)}{\mathrm{SO}(3,1)} \qquad Car_4 \cong \frac{Carr(4)}{\mathrm{ISO}(3)}.$$

- What does this have to do with time-like infinity?
- Analytic continuation of universal Ashtekar–Hansen structure at spatial infinity ("blow-up of spatial infinity") to time-like infinity



Ti as the blow-up of time-like infinity

To summarize, the result of the blowing up of i^0 is a 4-manifold which has the structure of a principal fibre bundle: The base space is the unit timelike hyperboloid in the tangent space of i^0, and the structure group is the additive group of reals. *This S will be called Spi—spatial infinity*. From its very construction, S inherits two tensor fields: a covariant, second rank, symmetric (degenerate) tensor field h_{ab}, the pullback to S of the natural metric on the hyperboloid $\mathcal{K}$; and a vertical vector field v^a, the

Outline

Carrollian fields on Ti

Question

What is a Ti field?
How do they encode Wigner's UIR of Poincaré?

Reminder:

- Wigner UIR $\phi^{\pm}_{a_1 \dots a_s}(x^a)$ of mass $m \neq 0$, spin s:
  - Defined on momentum orbit $p^\mu = \pm m(\cosh r, -\sinh r\, x^a/r)$;
  - transform under spin s-representation of SO(3).
- On Minkowski space, we like to work with Lorentz-covariant fields: $\phi_{\alpha_1 \dots \alpha_s}(x^\mu) = \phi_{(\alpha_1 \dots \alpha_s)}(x^\mu)$.
- Obtain UIR from fields by imposing field equations:

$$
\begin{aligned}
\left(\Box - m^2\right)\phi_{\alpha_1 \dots \alpha_s} &= 0\,, \\
\phi^{\alpha}{}_{\alpha\alpha_3 \dots \alpha_s} = \nabla^\alpha \phi_{\alpha\alpha_2 \dots \alpha_s} &= 0\,.
\end{aligned}
$$

- Find the analogous story on Ti!

Defining Ti fields

What is a Ti field?
Induce representations on Ti $\cong$ ISO(3, 1)/ ISO(3) from rep's of ISO(3).

- Choose a finite-component unitary representation $\phi_{a_1\dots a_s}(0)$ of $\mathfrak{iso}(3)$,

$$[J_{ab}, \phi_{a_1\dots a_s}(0)] = (\Sigma_{ab})_{a_1\dots a_s}^{b_1\dots b_s}\, \phi_{b_1\dots b_s}(0)\,, \qquad [P_a, \phi_{a_1\dots a_s}(0)] = 0\,,$$

  This defines the field transformation at the origin.

- Define the field on other points by (suppressing the indices)

$$\phi(x) \equiv \phi(\tau, \boldsymbol{x}) = U(\tau, \boldsymbol{x})\, \phi(0)\, U(\tau, \boldsymbol{x})^{-1} \qquad U(\tau, \boldsymbol{x}) = e^{x^a B_a} e^{\tau H}$$

- The induced infinitesimal action is then

$$\begin{aligned}
[J_{ab}, \phi(x)] &= -\xi^c_{J_{ab}} \partial_c \phi(x) + \Sigma_{ab}\, \phi(x)\,,\\
[B_a, \phi(x)] &= -\xi^c_{B_a} \partial_c \phi(x) - r^{-1} \tanh(r/2) x^b \Sigma_{ba}\, \phi(x)\,,\\
[H, \phi(x)] &= -\xi_H \phi(x) = \cosh r\, \partial_\tau \phi(x),\\
[P_a, \phi(x)] &= -\xi_{P_a} \phi(x) = -\sinh r\, x_a/r\, \partial_\tau \phi(x).
\end{aligned}$$

These transformations define Ti fields.

Ti fields and massive particles

What is the relation of these fields to Wigner's UIR?

- The representation is not irreducible. The Casimir $\mathcal{C}_2 = -H^2 + P^a P_a$ acts as

$$[\mathcal{C}_2, \phi(x)] = -\partial_\tau^2 \phi(x)\,,$$

  not proportional to the identity.

- Find irreducible representations by enforcing the Carrollian field equation

$$(\partial_\tau^2 + m^2)\phi_{a_1\ldots a_s}(x) = 0.$$

- Solutions:

$$\phi_m(\tau, \boldsymbol{x}) = \phi_m^+(\boldsymbol{x})\, e^{im\tau} + \phi_m^-(\boldsymbol{x})\, e^{-im\tau}\,.$$

  $\phi_m^\pm$: Wigner's UIR representation of $ISO(3,1)$ of mass m and spin s.

- "On-shell" Ti fields reduce to massive UIR of ISO(3,1).

Ti fields and massive UIR of ISO(3,1)

Ti fields and Wigner representations can also be related "off-shell".

- Start with a UIR $\phi_m^{\pm}(\boldsymbol{x})$ and define

$$\phi_m^{\pm}(\boldsymbol{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{\mp i m\tau}\phi(\tau,\boldsymbol{x})\,, \qquad \phi(\tau,\boldsymbol{x}) = \int_{-\infty}^{\infty} dm\, e^{\pm i m\tau}\phi_m^{\pm}(\boldsymbol{x})\,,$$

- Then, for any ISO(3,1) transformation T one has

$$[T^{\mathrm{Ti}},\phi(\tau,\boldsymbol{x})] = \int_{-\infty}^{\infty} dm\, e^{\pm i m\tau}[T^{\mathrm{Wigner}},\phi_m^{\pm}(\boldsymbol{x})]\,.$$

  Ti fields are superpositions of UIRs of different mass.

- The Ti representations are unitary w.r.t. the inner product

$$\langle\phi,\psi\rangle_{\mathrm{Ti}} = \frac{1}{2\pi}\int_{\mathrm{Ti}} \varepsilon\,\phi(\tau,\boldsymbol{x})\cdot\psi(\tau,\boldsymbol{x}), \qquad \varepsilon\ldots\text{inv. volume element on Ti}$$

- Denoting by $\langle\phi,\psi\rangle_m$ the inner product of Wigner UIR, one finds

$$\langle\phi,\psi\rangle_{\mathrm{Ti}} = \int_{-\infty}^{\infty} dm\,\langle\phi^{\pm},\psi^{\pm}\rangle_m\,.$$

Outline

Relation to fields in Minkowski space

Question

How do the so constructed Ti fields relate to fields in Minkowski space?

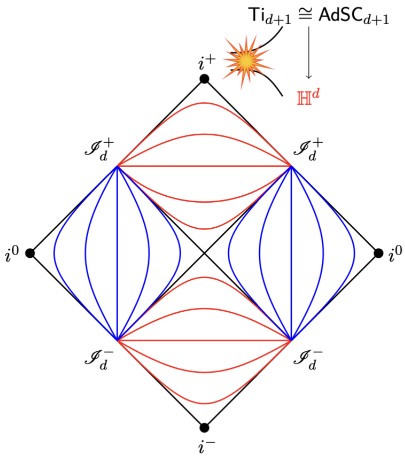
- Introduce hyperbolic coordinates

$$ds^2 = -d\tau^2 + \tau^2 h_{ij}(x^k)\, dx^i\, dx^j .$$

- Consider a symmetric tensor field obeying

$$(\Box - m^2)\phi_{\alpha_1\ldots\alpha_s} = 0\,,$$
$$\phi^{\alpha}{}_{\alpha\alpha_3\ldots\alpha_s} = \nabla^{\alpha}\phi_{\alpha\alpha_2\ldots\alpha_s} = 0\,.$$

- Project the field on temporal and spatial components; free data is in the purely spatial component $\phi_{i_1,\ldots i_s}$.



- The wave equation projected on the purely spatial components is

$$0 = \left[-\partial_\tau^2 + \tau^{-2}(\triangle_{\mathbb{H}} + c_1) - m^2\right](\tau^{3/2-s}\phi_{i_1\dots i_s}) + O(\tau^{-3}),$$

with leading order asymptotic solution $\phi_{i_1\dots i_s} = \tau^{-3/2+s}\bar{\phi}_{i_1\dots i_s}$ where

$$0 = (\partial_\tau^2 + m^2)\bar{\phi}_{i_1\dots i_s} \Rightarrow \qquad \bar{\phi}_{i_1\dots i_s} = \left(\phi^+_{i_1\dots i_s}(\boldsymbol{x})\, e^{im\tau} + \phi^-_{i_1\dots i_s}(\boldsymbol{x})\, e^{-im\tau}\right).$$

- Obeys the same equation as the Ti field. Does it transform the right way?
- Poincaré transformations in Cartesian coordinates become asymptotically Poincaré transformations in Ti representation

$$\{\partial_\alpha,\, x_\alpha\partial_\beta - x_\beta\partial_\alpha\} \qquad \overset{\tau\to\infty}{\to} \qquad \{\xi^{\mathrm{Ti}}_H,\, \xi^{\mathrm{Ti}}_{P_a},\, \xi^{\mathrm{Ti}}_{B_a},\, \xi^{\mathrm{Ti}}_{J_{ab}}\}\,.$$

- Introduce a suitable tetrad

$$E^A_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \tau e^a_i \end{pmatrix}, \qquad E^\alpha_A = \begin{pmatrix} 1 & 0 \\ 0 & \tau^{-1} e^i_a \end{pmatrix},$$

and define

$$\bar{\phi}_{a_1\dots a_s} := e^{i_1}_{a_1} \dots e^{i_s}_{a_s}\, \bar{\phi}_{i_1\dots i_s}$$

Ti extrapolate dictionary

This field transforms as

$$\delta\bar{\phi}_{a_1\ldots a_s} = -(\xi^\tau\partial_\tau + \xi^i\partial_i)\bar{\phi}_{a_1\ldots a_s} + \Lambda_{a_1}{}^b\,\bar{\phi}_{b\ldots a_s} + \ldots + \Lambda_{a_s}{}^b\,\bar{\phi}_{a_1\ldots b}\,.$$

This can be checked to match exactly with Ti transformation behavior!

Ti extrapolate dictionary

$$\bar{\phi}_{a_1\ldots a_s} \overset{\tau\to\infty}{\sim} \tau^{\frac{3}{2}} E^{\alpha_1}_{a_1}\ldots E^{\alpha_s}_{a_s}\phi_{\alpha_1\ldots\alpha_s}\,,$$

transforms as a Ti field and obeys the same Carrollian e.o.m.

Comments

- Compare with "extrapolate dictionary" for massless fields, e.g., scalar

$$\bar{\phi}(u,z,\bar{z}) = \lim_{r\to\infty}(r\phi(u,r,z,\bar{z})).$$

- Local interactions do not change this argument (long-range forces would).
- Can be also shown with saddle-point approximation of plane wave expansion

$$\phi_{\mu_1\ldots\mu_s}(X) = \sum_\sigma\int\frac{d^d\vec{k}}{2k^0(2\pi)^d}\left(\epsilon^\sigma_{\mu_1\ldots\mu_s}(k)\,a^\sigma_k\,e^{ik\cdot X} + \epsilon^\sigma_{\mu_1\ldots\mu_s}(k)^*\,a^{\sigma\,\dagger}_k\,e^{-ik\cdot X}\right).$$

Outline

Constructing a Ti EMT

Question

Can we construct conserved currents for the infinite number of symmetries of Ti?

Ingredients

- Carroll structure: $n = \partial_\tau$, $q = 0\, d\tau^2 + h_{ab}\, dx^a dx^b$
- Invariant volume form: $\varepsilon = \sqrt{h}\, d\tau\, d^3x$
- Symmetries: $\mathcal{L}_\xi n = \mathcal{L}_\xi q = 0 \qquad \Rightarrow \qquad \xi = S(x^a)\partial_\tau + \xi^a_{\mathbb{H}}\partial_a$
- Compatible (torsion-free) connection: $\nabla\, n = \nabla\, q = 0 \Rightarrow$not unique purely spatial part: $\nabla_{\mathbb{H}}$ connection on hyperboloid

Define EMT $T^\mu{}_\nu$ on Ti either by

- demanding that $j^\mu = T^\mu{}_\nu \xi^\nu$ is conserved

$$\nabla_\mu j^\mu = \nabla_\mu T^\mu{}_\nu \xi^\nu + T^\mu{}_\nu \nabla_\mu \xi^\nu = 0;$$

- or couple a Carrollian theory to arbitary curved background $S[\Phi_I; n^\mu, q_{\mu\nu}]$ compute $\delta S[\Phi_I; \delta n^\mu, \delta q_{\mu\nu}]$ and evaluate on Ti.

Both cases lead to Carrollian EMT of the form

$$T^{\mu}{}_{\nu} = \begin{pmatrix} T^{\tau}{}_{\tau} & 0 \\ T^{\tau}{}_{a} & T^{a}{}_{b} \end{pmatrix},$$

with T^{ab} symmetrical, that obeys the conservation equations

$$\begin{aligned} \nabla_\tau T^{\tau}{}_{\tau} &= 0 \qquad \Leftrightarrow \qquad \partial_\tau T^{\tau}{}_{\tau} = 0 \\ \nabla_\mu T^{\mu}{}_{a} &= 0 \qquad \Leftrightarrow \qquad \partial_\tau T^{\tau}{}_{a} + \nabla^{\mathbb{H}}_{b} T^{b}{}_{a} = 0\,, \end{aligned}$$

Define conserved charges as

$$Q[\xi] = \int_{\mathbb{H}} j \cdot \varepsilon = \int_{\mathbb{H}} \sqrt{h}\, d^3x\, j^\tau = \int_{\mathbb{H}} \sqrt{h}\, d^3x (T^{\tau}{}_{\mu} \xi^\mu)$$

In particular,

Ti charges

$$Q[S] = \int_{\mathbb{H}^3} \sqrt{h}\, d^3x\; T^{\tau}{}_{\tau} S \qquad Q[\xi^a_{\mathbb{H}}] = \int_{\mathbb{H}^3} \sqrt{h}\, d^3x\; T^{\tau}{}_{a} \xi^a_{\mathbb{H}}\,.$$

An example: Scalar field

Consider a massive scalar field in Minkowski space with possible local (polynomial) interactions and standard EMT $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\big(m^2\phi^2 + \partial_\mu\phi\partial^\mu\phi + V[\phi]\big)$.
Defining $\bar{\phi} = \tau^{3/2}\phi$ one has the leading order field equation

$$(\partial_\tau^2 + m^2)\bar{\phi} = 0 + \mathcal{O}(\tau^{-2}).$$

and setting $\bar{T}^\mu{}_\nu = \tau^3 T^\mu{}_\nu$ yields the leading order stress tensor

$$\bar{T}^\tau{}_\tau = -\frac{1}{2}\big((\partial_\tau\bar{\phi})^2 + m^2\bar{\phi}^2\big), \qquad \bar{T}^\tau{}_a = -\partial_\tau\bar{\phi}\,\partial_a\bar{\phi}, \qquad \bar{T}^a{}_b = \frac{1}{2}\delta^a_b\big((\partial_\tau\bar{\phi})^2 - m^2\bar{\phi}^2\big)\,,$$

of Carrollian form and obeying Carrollian conservation equations.
This is reproduced as e.o.m. and EMT of the electric scalar action on Ti

$$S_{\rm e} = -\frac{1}{2}\int d\tau\, d^3x\sqrt{h}\big[-n^\mu n^\nu\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} + m^2\bar{\phi}^2\big]\,.$$

Using the CCR $[\bar{\phi}(\tau,\boldsymbol{x}),\partial_\tau\bar{\phi}(\tau,\boldsymbol{x}')] = -i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')/\sqrt{h}$, one verifies

$$[Q[S],Q[S']] = 0, \qquad [Q[S],Q[\xi^a]] = -iQ[\xi^a\partial_a S] \qquad [Q[\xi_1],Q[\xi_2]] = iQ[\xi_{[1,2]}].$$

Asymptotically, the theory reduces to a simple Carrollian theory on Ti.

Conclusion and Outlook/Discussion

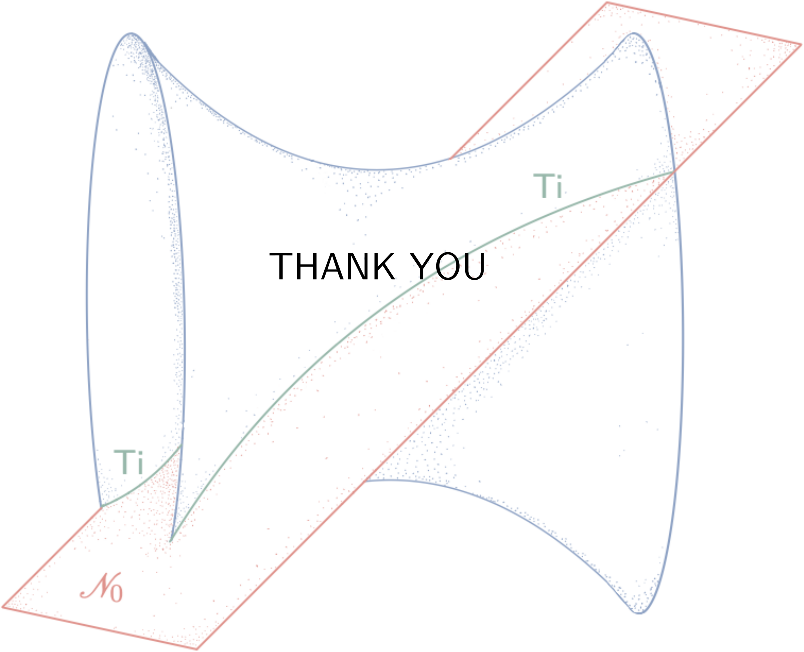
Summary

- Time-like infinity can be blown up to a Carrollian manifold Ti.
- Fields on Ti transform in a unitary, reducible representation of ISO(3,1).
- On-shell Ti fields reduce to Wigner's massive UIR times a phase.
- Massive fields in Minkowski space can be related to Ti fields as $\bar{\phi}_{a_1 \dots a_s} \overset{\tau \to \infty}{\sim} \tau^{\frac{3}{2}} E_{a_1}^{\alpha_1} \dots E_{a_s}^{\alpha_s} \phi_{\alpha_1 \dots \alpha_s}$.
- Conserved charges on Ti can be constructed using the Carrollian EMT.

Further developments and open issues

- Ti supertranslations and soft charges.
- Carrollian EMT and detector operators for massive particles.
- Changes in the presence of long-range forces (Maxwell, gravity).
- Generalization to fixed curved background.
- Reformulation of massive scattering amplitudes?
- A unifying Carrollian framework for massive and massless particles.

THANK YOU



Backup slides

Ti/AdS-Carroll algebra

- Carroll algebra

$$
\begin{aligned}
[J_{ab}, J_{cd}] &= \delta_{bc} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac} + \delta_{ad} J_{bc} \,, \\
[J_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b \,, & [C_a, P_b] &= \delta_{ab} H \,, \\
[J_{ab}, C_c] &= \delta_{bc} C_a - \delta_{ac} C_b \,,
\end{aligned}
$$

- Poincaré algebra

$$
\begin{aligned}
[J_{ab}, J_{cd}] &= \delta_{bc} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac} + \delta_{ad} J_{bc} \,, & [P_a, B_b] &= -\delta_{ab} H \,, \\
[J_{ab}, B_c] &= \delta_{bc} B_a - \delta_{ac} B_b \,, & [H, B_a] &= -P_a \,, \\
[J_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b \,, & [B_a, B_b] &= J_{ab} \,.
\end{aligned}
$$

- Let us relabel and reinterpret $P_a \to C_a, B_a \to P_a$

$$
\begin{aligned}
[J_{ab}, J_{cd}] &= \delta_{bc} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac} + \delta_{ad} J_{bc} \,, & [C_a, P_b] &= -\delta_{ab} H \,, \\
[J_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b \,, & [H, P_a] &= -C_a \,, \\
[J_{ab}, C_c] &= \delta_{bc} C_a - \delta_{ac} C_b \,, & [P_a, P_b] &= J_{ab} \,.
\end{aligned}
$$

  This is the Carroll algebra (changing $H \to -H$) in a curved space!
- Similar to interpretation of ISO$(3,1)$ as CCar_3.

Ti Supertranslations and soft charges

- In an interacting theory of gravity, truly conserved charges reside at spatial infinity. In particular, supertranslations are conserved.
- Using the flux balance laws this charge $Q(i^0)$ can be rewritten as

$$Q(i^0) = Q(i^+) - F(\mathscr{I}^+) = Q(i^-) + F(\mathscr{I}^-)\,.$$

- Using invariance of the S-matrix $[Q(i^0), \mathcal{S}]=0$ and evaluating between in- and out-states leads to Weinberg's soft graviton theorem.
- $Q(i^+)$ in the above is labelled by a function on the sphere $f(\Omega)$.
- Can be related to Ti charge as

$$S(\boldsymbol{x}) = \int_{\mathbb{CS}^2} d\Omega\, G(\boldsymbol{x}, \Omega) f(\Omega)\,,$$

where $G(\boldsymbol{x}, \Omega)$ is a 'boundary-to-bulk' propagator satisfying the gauge-fixing equation

$$(\triangle_{\mathbb{H}} - 3)\, G(\boldsymbol{x}) = 0\,,$$

and the boundary value of $S(\boldsymbol{x})$ is $f(\Omega)$.

EMT algebra and detector operators

Detector operators describe possible measurements of weighted cross-sections; important in QCD, more recently CFTs.

- For massless particles, described as integrals over $\mathscr{I}^+$; important example: ANEC operator

$$\mathcal{E}(\mathbf{n}) = \int_0^\infty dt \lim_{r\to+\infty} r^2 n^i T_{0i}(t, r\mathbf{n}).$$

- For massive particles, possible detectors and their algebra can be understood in terms of the Carrollian EMT and its algebra

$$[\bar{T}^\tau{}_\tau(\boldsymbol{y}), \bar{T}^\tau{}_\tau(\boldsymbol{y}')] = 0$$

$$[\bar{T}^\tau{}_\tau(\boldsymbol{y}), \bar{T}^\tau{}_a(\boldsymbol{y}')] = i\big(\partial_a \bar{T}^\tau{}_\tau(\boldsymbol{y}) - \bar{T}^\tau{}_\tau(\boldsymbol{y})\partial'_a + \bar{T}^b{}_a(\boldsymbol{y})\partial'_b\big)\,\hat{\delta}^{(d)}(\boldsymbol{y}-\boldsymbol{y}') \ldots$$

Hallmark of Carrollian symmetry.

- Consider the final state of n massive scalar particles

$$|n\rangle = |p_1, \ldots, p_n\rangle = a^\dagger_{p_1} \ldots a^\dagger_{p_n} |0\rangle = \left(m^{-1/2} 2(2\pi)^{d/2}\right)^n \phi^-(\boldsymbol{y}_1) \ldots \phi^-(\boldsymbol{y}_n) |0\rangle \, .$$

- One can construct, e.g., an operator that measure the total final energy

$$-\int_{\mathbb{H}} \varepsilon^{(d)} :T^\tau{}_\tau(\boldsymbol{y}): |n\rangle = n\, m\, |n\rangle \ ,$$

- The analogue of the energy flow operator is

$$\mathcal{E}(\boldsymbol{n}) \equiv -\int_0^\infty d\|\boldsymbol{y}\| \|\boldsymbol{y}\|^{d-1} :T^\tau{}_\tau(\|\boldsymbol{y}\|\boldsymbol{n}): \ .$$

- Applied to a final state this is

$$\mathcal{E}(\boldsymbol{n}) |n\rangle = \sum_{i=1}^n E_i\, \delta^{(d-1)}(\boldsymbol{n} - \boldsymbol{n}_i) |n\rangle \ .$$

- The important commutator

$$[\mathcal{E}(\boldsymbol{n}), \mathcal{E}(\boldsymbol{n}')] = 0 \, ,$$

follows directly from the hallmark equation of Carrollian symmetry.

Long-range forces

Massive particles coupled to long-range forces are not free even at $\tau \to \infty$.
Consider, e.g., massive scalar QED at late time the field is

$$\phi \overset{\tau\to\infty}{\to} \tau^{-3/2} e^{ieA_\tau \log\tau}(b(\boldsymbol{x})e^{-i\tau m} + d^*(\boldsymbol{x})e^{+i\tau m}).$$

This leads to the Ti field

$$\bar{\phi} = e^{ieA_\tau \log\tau}(b(\boldsymbol{x})e^{-i\tau m} + d^*(\boldsymbol{x})e^{+i\tau m}),$$

not of the free field form $(\partial_\tau^2 + m^2)\bar{\phi} = 0$ but still Ti representation.
Could be interesting to:

- Use the relation between UIR and Ti field to understand it as a superposition over UIRs

$$\phi_m^\pm(\boldsymbol{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{\mp im\tau}\phi(\tau,\boldsymbol{x})\,, \qquad \phi(\tau,\boldsymbol{x}) = \int_{-\infty}^{\infty} dm\, e^{\pm im\tau}\phi_m^\pm(\boldsymbol{x})\,,$$

- Understand $\bar{\phi}$ as solution of Carrollian massive field coupled to (magnetic) Carrollian EDyn on Ti.

Wigner's massive UIR

For Wigner's massive UIR, the isotropy group is $\langle J_{ab}, H, P_a \rangle$. The inducing representation is given by

$$\left[J_{ab}, \phi^{\pm}_{a_1 \dots a_s}(\mathbf{0})\right] = (\Sigma_{ab})_{a_1 \dots a_s}^{b_1 \dots b_s}\, \phi^{\pm}_{b_1 \dots b_s}(\mathbf{0})\,, \qquad \left[P_a, \phi^{\pm}(\mathbf{0})\right] = 0\,,$$

and

$$\left[H, \phi^{\pm}(\mathbf{0})\right] = \pm i m\, \phi(\mathbf{0})\,,$$

corresponding to a massive particle in the rest frame with momentum $p^{\pm}_{\mu} = \pm(m, \mathbf{0})$. The induced representation is then obtained by boosting $\phi^{\pm}$ to a generic momentum frame,

$$\phi^{\pm}(\boldsymbol{x}) = U(\boldsymbol{x})\, \phi^{\pm}(\mathbf{0})\, U(\boldsymbol{x})^{-1}\,,$$

where $U(\boldsymbol{x}) = e^{x^a B_a}$. This implies

$$\begin{aligned}
[J_{ab}, \phi^{\pm}(\boldsymbol{x})] &= -\xi_{J_{ab}} \phi^{\pm}(\boldsymbol{x}) + \Sigma_{ab}\, \phi^{\pm}(\boldsymbol{x})\,,\\
[B_a, \phi^{\pm}(\boldsymbol{x})] &= -\xi_{B_a} \phi^{\pm}(\boldsymbol{x}) - \tanh(r/2) \hat{x}^b \Sigma_{ba}\, \phi^{\pm}(\boldsymbol{x})\,,\\
[H, \phi^{\pm}(\boldsymbol{x})] &= \pm i m \cosh(r)\, \phi^{\pm}(\boldsymbol{x})\,,\\
[P_a, \phi^{\pm}(\boldsymbol{x})] &= \mp i m \sinh(r)\, \hat{x}_a\, \phi^{\pm}(\boldsymbol{x})\,.
\end{aligned}$$