Dual Gravity ⓪

- Dirac thought that Maxwell's equations should have a duality symmetry

$$E_i \leftrightarrow B_i$$

- Montonen and Olive proposed a duality symmetry between electric particles and magnetic monopoles. Realized in the maximally supersymmetric Yang-Mills theory

- E theory unifies all maximal supergravity theories and so all type II superstring theory at low energy. E_{11} contains a vast duality symmetry

- Does gravity have a duality symmetry.

Dual Gravity

The history

Curtwright (1985) noticed that the field $A_{\mu\nu,\lambda}$ in 5 dimensions had the degrees of freedom of gravity

Hull (2000) suggested that $A_{\mu_1 \ldots \mu_{D-3},\nu}$ could describe dual gravity in D dimensions hep-th/0004195

West (2001) hep-th/0104081

The non-linear realization of E_{11} had the fields

$$h_a{}^b,\ A_{a_1 a_2 a_3},\ A_{a_1 \ldots a_6},\ h_{a_1 \ldots a_8, b}, \ \ldots$$

(may be dual graviton).

We can write Einstein gravity in D dimensions as

$$\int d^Dx\, e R = \int d^D x\, e \left(-\frac{1}{4} C_{\mu\nu,\rho} C^{\mu\nu,\rho} - \frac{1}{2} C_{\mu\nu,\rho} C^{\mu\rho,\nu} + C_{\mu\rho,}{}^{\rho} C^{\mu\tau}{}_{,\tau} \right)$$

where $C_{\mu\nu,\rho} = (\partial_\mu e_\nu^a - \partial_\nu e_\mu^a) e_{a\rho}$

This is equivalent to

$$\frac{1}{2} \int d^Dx\, e \left(\hat{Y}^{\mu\nu,\rho} C_{\mu\nu,\rho} + \frac{1}{2} \hat{Y}_{\mu\nu,\rho} \hat{Y}^{\mu\rho,\nu} - \frac{1}{2(D-2)} \hat{Y}_{\mu\rho,}{}^{\rho} \hat{Y}^{\mu\tau}{}_{,\tau} \right)$$

we can instead use

$$\hat{Y}^{\mu_1\mu_2,\rho} = \frac{1}{(D-2)!} \epsilon^{\mu_1\mu_2\nu_1 \ldots \nu_{D-2}} Y_{\nu_1 \ldots \nu_{D-2},}{}^{\rho}$$

Then the action has the form

$$\frac{1}{2}\int d^Dx\left(\epsilon^{\mu\nu\tau_1\ldots\tau_{D-2}}\, Y_{\tau_1\ldots\tau_{D-2},}{}^{\rho}\, C_{\mu\nu,\rho} + e\, Y^2 \text{ terms}\right)$$

The equations of motion are

$$\epsilon^{\mu\tau_1\ldots\tau_{D-1}}\,\partial_{\tau_1} Y_{\tau_2\ \ \tau_{D-1}}{}^{\rho} = Y^2 \text{ terms}$$

$$\epsilon_{\mu\nu}{}^{\tau_1\ldots\tau_{D-2}}\, Y_{\tau_1\ldots\tau_{D-2},\rho} = -C_{\mu\nu,\rho} + C_{\nu\rho,\mu} - C_{\mu\rho,\nu} + 2\left(g_{\nu\rho}\, C_{\mu,\tau}{}^{\tau} - g_{\mu\rho}\, C_{\nu,\tau}{}^{\tau}\right)$$

at the linearized level

$$Y_{\tau_1\ldots\tau_{D-2},\rho} = \partial_{[\tau_1} h_{\tau_2\ldots\tau_{D-2}],\rho}$$

Hence $h_{\tau_1\ldots\tau_{D-3},\rho}$ does describe gravity at the linearized level with an action deduced from above.

Problem E_{11} had $h_{a_1\ldots a_8,b}$ subject to $h_{[a_1\ldots a_8,b]} = 0$ ie an irreducible field.

It is missing $h_{a_1\ldots a_9}$.

3.

Linearized gravity and dual gravity unfolded.

Gravity field $h_{\mu\nu}$ with spin connection $\omega_{\mu,\nu_1\nu_2}$
$= -\omega_{\mu,\nu_2\nu_1}$

$$\partial_{[\mu_1} h_{\mu_2]}{}^{\nu} + \omega_{[\mu_1,\mu_2]}{}^{\nu} = 0$$

invariant under

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \Lambda_{\mu\nu}, \delta\omega_{\mu,\nu_1\nu_2} = -\partial_\mu \Lambda_{\nu_1\nu_2}$$

with $\Lambda_{\mu_1\mu_2} = -\Lambda_{\mu_2\mu_1}$

Invariant is

$$R_{\mu_1\mu_2,}{}^{\nu_1\nu_2} = \partial_{[\mu_1} \omega_{\mu_2],}{}^{\nu_1\nu_2}$$

with equation of motion $R_{\mu\rho,}{}^{\rho\nu} = 0$

Can use $\Lambda_{\mu\nu}$ to set $h_{[\mu\nu]} = 0$

Solving for $\omega_{\mu,\nu_1\nu_2}$ we find

$$\omega_{\mu,\nu_1\nu_2} = -\partial_{\nu_1} \overset{S}{h}_{\nu_2\mu} + \partial_{\nu_2} \overset{S}{h}_{\nu_1\mu} + \partial_\mu \overset{A}{h}_{\nu_1\nu_2}$$

and so

$$R_{\mu_1\mu_2,}{}^{\nu_1\nu_2} = \partial^{[\nu_1} \partial_{[\mu_1} h_{\mu_2]}{}^{\nu_2]}$$

Dual gravity

field $h_{\mu_1 \cdots \mu_{D-3},\rho}$ connection $\omega_{\mu,\nu_1 \cdots \nu_{D-2}}$ defined by

$$\partial_{[\mu_1} h_{\mu_2 \cdots \mu_{D-2}],}{}^{\nu} + \omega_{[\mu_1,\mu_2 \cdots \mu_{D-2}]}{}^{\nu} = 0$$

Invariant under

$$\delta h_{\mu_1 \cdots \mu_{D-3},\nu} = \partial_{[\mu_1} \Lambda_{\mu_2 \cdots \mu_{D-3}],\nu} + \hat{\Lambda}_{\mu_1 \cdots \mu_{D-3}\nu}$$

$$\delta \omega_{\mu,\nu_1 \cdots \nu_{D-2}} = -\partial_\mu \hat{\Lambda}_{\nu_1 \cdots \nu_{D-2}}.$$

An invariant is

$$R^{\nu_1 \cdots \nu_{D-2}}{}_{,\mu_1\mu_2} = \partial_{[\mu_1} \omega_{\mu_2],}{}^{\nu_1 \cdots \nu_{D-2}}$$

and the equation of motion is

$$R^{\lambda \nu_1 \cdots \nu_{D-3}}{}_{,\lambda\mu} = 0$$

The fields and parameters are reducible

$$h_{\mu_1 \cdots \mu_{D-3},\nu} = h^{I}_{\mu_1 \cdots \mu_{D-3},\nu} + \hat{h}^{I}_{\mu_1 \cdots \mu_{D-3}\nu}$$

$$\Lambda_{\nu_1 \cdots \nu_{D-4},\lambda} = \Lambda^{I}_{\nu_1 \cdots \nu_{D-4},\lambda} + \hat{\Lambda}^{I}_{\nu_1 \cdots \nu_{D-4}\lambda}$$

where $h^{I}_{[\mu_1 \cdots \mu_{D-3},\nu]} = 0 = \Lambda^{I}_{[\nu_1 \cdots \nu_{D-4},\lambda]}$

Then

$$\delta h^I_{\mu_1 \cdots \mu_{D-3}, \nu} = \partial_{[\mu_1} \Lambda^I_{\mu_2 \cdots \mu_{D-3}], \nu} - \frac{1}{9} \partial_\nu \tilde{\Lambda}^I_{\mu_1 \cdots \mu_{D-3}} + \frac{1}{9} \partial_{[\mu_1} \tilde{\Lambda}^I_{\mu_2 \cdots \mu_{D-3}]\nu}$$

$$\delta h^I_{\mu_1 \cdots \mu_{D-2}} = \partial_{[\mu_1} \tilde{\Lambda}^I_{\mu_2 \cdots \mu_{D-2}]} + \hat{\Lambda}_{\mu_1 \cdots \mu_{D-2}}$$

The spin connection is

$$\omega_{\mu, \nu_1 \cdots \nu_{D-2}} = -9\, \partial_{[\nu_1} h^I_{\nu_2 \cdots \nu_{D-2}], \mu} - \partial_\mu h^I_{\nu_1 \cdots \nu_{D-2}}.$$

Can use $\hat{\Lambda}_{\mu_1 \cdots \mu_{D-2}}$ to set $h^I_{\nu_1 \cdots \nu_{D-2}} = 0$

leaving the irreducible $h^I_{\nu_1 \cdots \nu_{D-3}, \mu}$

Either way the field equation is

$$E_{\mu_1 \cdots \mu_{D-3},}{}^{\nu} \equiv \partial^{[\lambda} \partial_{[\mu_1} h^I_{\mu_2 \cdots \mu_{D-3} \lambda],}{}^{\nu]} = 0$$

It only involves the irreducible field.

$h^I_{\mu_1 \cdots \mu_{D-3}, \nu}$ and $E_{(\nu_1 \cdots \nu_{D-3}, \lambda)} = 0$

The non-linear dual ^{6} graviton

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hep-th 2006 .02383

obeys in $D=11$ the equation

$$E_{\mu_1\ldots\mu_8,\tau} \equiv g^{\nu\kappa}\partial_{[\nu|}F_{[\kappa,\mu_1\ldots\mu_8],|\tau]} - \frac{1}{9}g^{\nu\kappa}\hat{G}_{\tau,\rho}{}^{\rho}\hat{G}_{[\mu_1,\mu_2\ldots\mu_8]\nu,\kappa} - \frac{1}{9}g^{\nu\kappa}\hat{G}_{[\mu_1|,\rho}{}^{\rho}\hat{G}_{\tau,|\mu_2\ldots\mu_8]\nu,\kappa}$$
$$+\frac{1}{2}g^{\nu\kappa}\hat{G}_{\nu,\rho}{}^{\rho}\hat{G}_{[\kappa,\mu_1\ldots\mu_8],\tau} - \frac{1}{2\cdot 9}g^{\nu\kappa}\hat{G}_{\nu,\rho}{}^{\rho}\hat{G}_{\tau,\mu_1\ldots\mu_8,\kappa} - \hat{G}_{\nu,}{}^{(\kappa\nu)}\hat{G}_{[\kappa,\mu_1\ldots\mu_8],\tau}$$
$$+\frac{1}{9}\hat{G}_{\nu,}{}^{(\kappa\nu)}\hat{G}_{\tau,\mu_1\ldots\mu_8,\kappa} + \frac{4}{9}\hat{G}_{\tau,}{}^{(\nu\kappa)}\hat{G}_{[\mu_1,\mu_2\ldots\mu_8]\nu,\kappa} + \frac{4}{9}\hat{G}_{[\mu_1|,}{}^{(\nu\kappa)}\hat{G}_{\tau,|\mu_2\ldots\mu_8]\nu,\kappa}$$
$$+(\det e)^{-1}\varepsilon^{\kappa_1\kappa_2\nu_1\ldots\nu_9}\hat{G}_{\nu_1,\nu_2\ldots\nu_9,[\mu_1|}\hat{G}_{[\kappa_1,\kappa_2|\mu_2\ldots\mu_8]],\tau} = 0$$

where

$$F_{\kappa,\mu_1\ldots\mu_8,\tau} = \partial_{[\kappa} h^{I}_{\mu_1\ldots\mu_8],\tau}$$

$$\hat{G}_{\mu,\nu}{}^{\rho} = \partial_\mu e_\nu{}^c e_c{}^\rho$$

- It contains the irreducible field $h^{I}_{\mu_1\ldots\mu_8,\tau}$ and $e_\mu{}^a$
- $E_{[\mu_1\ldots\mu_8,\tau]} = 0$
- It agrees with the linearized result
- Derived from E_{11} by varying the six form equation of motion which in turn came from the three form equation of motion. Set the 3 and 6 form to zero
- Is gauge invariant

E_{11} also gives a gravity - dual gravity relation by varying the 3-6 form duality relation. At the linearised level

West hep-th 1411.0920

$$\mathcal{E}_{\mu,}{}^{L}{}^{\nu_1\nu_2} = \omega_{\mu,}{}^{\nu_1\nu_2} - \frac{1}{4}\epsilon^{\nu_1\nu_2\rho_1\cdots\rho_9}\partial_{\rho_1}h^{I}_{\rho_2\cdots\rho_9,\mu} \doteq 0 \qquad (1)$$

Taking the trace

$$\omega_{\mu,}{}^{\nu\mu} \doteq \frac{1}{4}\epsilon^{\nu\rho_1\cdots\rho_{10}}\partial_{\rho_1}h^{I}_{\rho_2\cdots\rho_9,\rho_{10}} = 0 \; !$$

But (1) is not invariant

$$\delta\,\mathcal{E}^{L}_{\mu,}{}^{\nu_1\nu_2} = \partial_\mu \hat{\lambda}^{\nu_1\nu_2}$$

We must think of it as an equivalence relation

$$\mathcal{E}^{L}_{\mu,}{}^{\nu_1\nu_2} \sim \mathcal{E}^{L}_{\mu,}{}^{\nu_1\nu_2} + \partial_\mu\hat{\lambda}^{\nu_1\nu_2}$$

By taking a derivative we can find a normal equation of motion and eliminate one field

$$\partial_{[\mu}\omega_{\lambda],}{}^{\nu\lambda} = \frac{1}{4}\epsilon^{\nu\lambda\rho_1\cdots\rho_9}\partial_{[\mu}\partial_{\rho_1}h^{I}_{\rho_2\cdots\rho_9,\lambda]} = 0$$

Similarly

$$\partial^{\alpha}\partial_{[\alpha}h^{I}{}_{\rho_2\cdots\rho_{D-2}],}{}^{\mu]} = 0.$$

E_{11} has duality relations which are equivalence relations but field equations which are normal equations

Duality relations $\Longleftrightarrow$ field equations

Alternatively one can add $h_{\mu_1 \cdots \mu_9}$ as we did before. Then

$$\omega_{\mu,}{}^{\nu_1 \nu_2} - \frac{1}{4} \epsilon^{\nu_1 \nu_2 \rho_1 \cdots \rho_9} \left(\partial_{\rho_1} h^{I}_{\rho_2 \cdots \rho_9}{}^{\mu} + \partial_{\mu} h_{\rho_1 \cdots \rho_9} \right) = 0$$

Kac Moody A_1^{+++} and gravity

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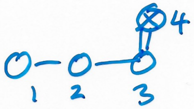

Deleting node 4 gives $GL(4)$ and so a "four" dimensional theory

The fields in the A_1^{+++} non-linear realization

h_{ab} gravity, $\tilde{h}_{ab}$ dual gravity, $\tilde{\tilde{h}}_{[a_1a_2],(b_1b_2)}$ dual-dual gravity, ...

The coordinates

$$x^\mu, y^\mu, \dots$$

The non-linear realization has invariant equations of motion

- Einstein equation for gravity h_{ab}
- The non-linear dual gravity equation
- a gravity-dual gravity duality

relation

The dual graviton equation is

$$\overline{E}_{\mu\nu} = g^{\rho\sigma}\partial_{[\sigma|}\overline{F}_{[\rho,\nu]|\mu]} + \frac{1}{4}g^{\rho\sigma}G_{\tau,\rho}{}^{\tau}(\overline{G}_{\nu,\mu\sigma} + \overline{G}_{\mu,\sigma\nu} - \overline{G}_{\sigma,\mu\nu})$$

$$+\frac{1}{4}g^{\rho\sigma}G_{\rho,\tau}{}^{\tau}(-\overline{G}_{\nu,\mu\sigma} - \overline{G}_{\mu,\nu\sigma} + \overline{G}_{\sigma,\mu\nu}) + \frac{1}{4}g^{\rho\sigma}G_{\rho,\sigma}{}^{\tau}(-\overline{G}_{\tau,\mu\nu} + \overline{G}_{\mu,\nu\tau} + \overline{G}_{\nu,\mu\tau})$$

$$-\frac{1}{4}g^{\rho\sigma}G_{\nu,\rho}{}^{\tau}\overline{G}_{\mu,\tau\sigma} - g^{\rho\sigma}\frac{1}{4}G_{\mu,\rho}{}^{\tau}\overline{G}_{\nu,\tau\sigma} + \frac{1}{16}g^{\rho\sigma}G_{\nu,\tau}{}^{\tau}\overline{G}_{\mu,\rho\sigma} + \frac{1}{16}g^{\rho\sigma}G_{\mu,\tau}{}^{\tau}\overline{G}_{\nu,\rho\sigma}$$

$$-\frac{1}{4}(\det e)^{-1}\varepsilon^{\tau_1\tau_2\tau_3\tau_4}\overline{G}_{[\tau_1,\tau_2]\mu}\overline{G}_{[\tau_3,\tau_4]\nu}$$

where

$$\overline{F}_{\mu,\nu_1\nu_2} = \partial_\mu \hat{h}_{\nu_1\nu_2}$$

$$G_{\mu,\nu}{}^{\rho} = \partial_\mu e_\nu{}^{c}\, e_c{}^{\rho}$$

The gravity - dual gravity duality relation is

$$w_{a,b_1b_2} + \frac{1}{2}\epsilon_{b_1b_2}{}^{c_1c_2}\,\overline{G}_{c_1,c_2a} \doteq 0$$

where

$$\overline{G}_{c_1,d_1d_2} = e_{c_1}{}^{\mu}\, e_{d_1}{}^{\nu_1} e_{d_2}{}^{\nu_2}\,\partial_\mu A_{\nu_1\nu_2}.$$

In the non-linear realization of E_{11} has a cosmological constant in D dimensions which arises from D-1 forms $A_{\mu_1 \dots \mu_{D-1}}$

Field strength $F_{\mu_1 \dots \mu_D}$ with action

$$\int d^D x \, e \, F_{\mu_1 \dots \mu_D} F^{\mu_1 \dots \mu_D}.$$

Equation of motion

$$\partial_{\mu_1} F^{\mu_1 \nu_1 \dots \nu_{D-1}} = 0$$

$\rightarrow F_{\mu_1 \dots \mu_D} = m \, \epsilon_{\mu_1 \dots \mu_D}$ with action

$$\int d^4 x \, e \, m^2$$

West hep-th/2007.11925.

There is no three form in A_1^{+++} $\rightarrow$ no cosmological constant in four dimensions. If the duality symmetry is spontaneously broken we can have a cosmological constant.

The moral

E_{11} knows best.