Matter Coupled Carroll Gravity

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based upon work in progress done with P. Concha, E. Rodríguez and O. Fierro

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Outline

Carroll Scalars

Outline

Outline

Outline

Outline

Outline

Outline

Electric Carroll Scalars

Consider the following Lagrangian describing a 4D relativistic real scalar Φ :

$$\mathcal{L} = \frac{\tilde{c}^2}{2}(\partial_t\Phi)^2 - \frac{1}{2}\partial_a\Phi\partial^a\Phi \; - \; \frac{M^2}{2\tilde{c}^2}\Phi^2 \,, \qquad\qquad a = 1,2,3$$

with mass M and $\tilde{c} \equiv 1/c$. Making the redefinitions

$$\Phi = \frac{\phi}{\tilde{c}} \,, \qquad\qquad M = m\tilde{c}^2$$

and taking $\tilde{c} \to \infty$, we obtain the following electric Carroll scalar Lagrangian :

$$\mathcal{L}_{\text{electric scalar}} = \frac{1}{2}\,(\partial_t\phi)^2 - \frac{m^2}{2}\phi^2$$

Under Carroll boosts : $\partial_a\phi \;\;\to\;\; \partial_t\phi \;\;\to\;\; 0$

The electric Carroll particle has non-zero energy but cannot move

Magnetic Carroll Scalars

de Boer, Hartong, Obers, Sybesma, Vandoren (2021); Henneaux, Salgado-Rebolledo (2021)

We start from the same Lagrangian as in the electric case but written in Hamiltonian form, introducing an auxiliary field Π, and with an opposite sign of the mass term :

$$\mathcal{L} = \Pi\partial_t\Phi - \frac{1}{2\tilde{c}^2}\Pi^2 - \frac{1}{2}\partial_a\Phi\partial^a\Phi + \frac{M^2}{2\tilde{c}^2}\Phi^2$$

Making the redefinitions

$$\Pi = \pi\,, \qquad \Phi = \phi\,, \qquad M = m\tilde{c}$$

and taking $\tilde{c} \to \infty$ we obtain the following magnetic Carroll scalar Lagrangian :

$$\mathcal{L}_{\text{magnetic scalar}} = \pi\partial_t\phi - \frac{1}{2}\partial_a\phi\partial^a\phi + \frac{m^2}{2}\phi^2$$

under Carroll boosts: $\pi \quad \to \quad \partial_a\phi \quad \to \quad \partial_t\phi \quad \to 0$

The magnetic Carroll particle can move but has zero energy

When do we have moving Carroll particles ?

- considering two interacting Carroll particles or coupling to background fields

  Gomis, Longhi + E.B. (2014), see talk by Gomis

- taking a magnetic limit

  de Boer, Hartong, Obers, Sybesma, Vandoren (2021); Henneaux, Salgado-Rebolledo (2021)

- giving up Carroll boosts

  Ciambelli (2023)

- considering two interacting particles using field theory

  Ecker, Grumiller, Henneaux, Salgado-Rebolledo (2024)

  see talk by Henneaux

An Ultratension Limit of String Theory

Carroll tensionless limit

Isberg, Lindström, Sundborg, Theodoridis (1994); Bagchi, Chakrabortty, Parekh (2016)
Bidussi, Harmark, Hartong, Obers, Oling (2023)

Taking a worldsheet electric Carroll limit of the Polyakov action together with $T = T_e/\tilde{c}$, we obtain in the limit that $\tilde{c} \to \infty$:

$$S_{\text{tensionless}} = -T_e \int d^2\xi \, e \, \tau^\alpha \tau^\beta \, \partial_\alpha \mathbf{X} \cdot \partial_\beta \mathbf{X}$$

We have effectively sent

$$T \to 0 \,, \quad \tilde{c} \to \infty \,, \quad \text{with} \qquad T_e \equiv \tilde{c} T \text{ fixed}$$

Carroll ultratension limit

cp. to Blair, Lahnsteiner, Obers, Yan (2023); Hohm, Townsend + E.B., work in progress

Taking a worldsheet magnetic Carroll limit together with $T = \tilde{c} T_m$, we obtain

$$S_{\text{ultratension}} = T_m \int d^2\xi \, e \big(\boldsymbol{\pi} \cdot \tau^\alpha \partial_\alpha \mathbf{X} + e_1{}^\alpha e_1^\beta \, \partial_\alpha \mathbf{X} \cdot \partial_\beta \mathbf{X} \big)$$

where we have been sending

$$T \to \infty \,, \quad \tilde{c} \to \infty \,, \quad \text{with} \qquad T_m \equiv \frac{T}{\tilde{c}} \text{ fixed}$$

Outline

Cartan Formulation of Lorentzian Geometry

Figueroa-O'Farrill, van Helden, Rosseel, Rotko, ter Veldhuis (2023)

The solder form ${E_\mu}^A$ transforms under infinitesimal local Lorentz transformations with parameters $\Lambda^{AB} = -\Lambda^{BA}$ as

$$\delta {E_\mu}^A = -{\Lambda^A}_B {E_\mu}^B$$

A metric-compatible spin-connection ${\Omega_\mu}^{AB} = -{\Omega_\mu}^{BA}$ with torsion ${T_{\mu\nu}}^A$ satisfies the following first Cartan structure equations:

$${T_{\mu\nu}}^A = 2\partial_{[\mu} {E_{\nu]}}^A - 2{\Omega_\mu}^{AB} E_{\nu]b}$$

In Riemannian geometry we have

1. All spin-connection components can be solved for in terms of the Vierbeine ${E_\mu}^A$ and the torsion tensors ${T_{\mu\nu}}^A$:

$${\Omega_\mu}^{AB} = {\Omega_\mu}^{AB}(E, T)$$

2. Each torsion tensor component contains a spin-connection field

Carroll Geometry as a Limit

Using a first-order formulation, the Carroll solder forms $(\tau_\mu\,, e_\mu{}^a)$ and Carroll spin-connections $(\omega_\mu{}^{ab}\,, \omega_\mu{}^{0a})$ can be obtained by making the following redefinitions:

$$E_\mu{}^0 = \frac{1}{\tilde{c}}\tau_\mu\,, \qquad E_\mu{}^a = e_\mu{}^a\,, \qquad T_{\mu\nu}{}^a = t_{\mu\nu}{}^a\,,$$

$$\Omega_\mu{}^{ab} = \omega_\mu{}^{ab}\,, \qquad \Omega_\mu{}^{a0} = \frac{1}{\tilde{c}}\omega_\mu{}^{a0}\,, \qquad T_{\mu\nu}{}^0 = t_{\mu\nu}{}^0$$

After taking $\tilde{c} \to \infty$ in the first Cartan structure equations we find that [1]

1. The torsion tensor components $t_{0(a,b)}$ do not contain any spin-connection component
2. The spin-connection components $\omega^{(a,0b)}$ can not be solved for in terms of the solder forms $(\tau_\mu\,, e_\mu{}^a)$ and the torsion tensors $t_{\mu\nu}{}^0\,, t_{\mu\nu}{}^a$

The torsion tensor components $t_{0(a,b)}$ are called intrinsic torsion tensors. Setting them to zero leads to geometric constraints

[1] We define $X_0 \equiv \tau^\mu X_\mu\,, \quad X_a \equiv e_a{}^\mu X_\mu\,, \qquad X_{0a} \equiv \tau^\mu e_a{}^\nu X_{\mu\nu}\,, \quad X_{ab} \equiv e_a{}^\mu e_b{}^\nu X_{\mu\nu}$

Four Carroll Geometries

Figueroa-O'Farrill (2020)

Setting boost-invariant combinations of the intrinsic torsion tensors equal to zero leads to four Carroll geometries :

Carroll 1: all intrinsic torsion tensors are non-zero : no constraints

Carroll 2: $t_{0a,}{}^{a} = 0$: the 3-form $\Omega \equiv \epsilon_{abc}\, e_\mu{}^a e_\nu{}^b e_\rho{}^c$ is closed : $d\Omega = 0$

Carroll 3: $t_{0\{a,b\}} = 0$: the vector τ^μ is a conformal Killing vector with respect to the spatial metric $h_{\mu\nu} = e_\mu{}^a e_\nu{}^b \delta_{ab}$ or zero extrinsic curvature

Carroll 4: all intrinsic torsion tensors are zero : both constraints

Outline

Magnetic Carroll Gravity

Gomis, Rollier, Rosseel, ter Veldhuis + E.B. (2017)

Taking the Carroll limit, along with $G_N = G_C/\tilde{c}$, of the EH action with zero torsion

$$S_{\text{1st-order EH}} = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\, E E_A{}^\mu E_B{}^\nu R_{\mu\nu}{}^{AB}(\Omega)$$

leads to the following first-order magnetic Carroll gravity action:

$$S_{\text{1st order Carroll grav.}} = \frac{1}{16\pi G_C}\int \mathrm{d}^4x\, e\Big(e_a{}^\mu e_b{}^\nu R_{\mu\nu}(J)^{ab} + 2\tau^\mu e_a{}^\nu R_{\mu\nu}(G)^{0a}\Big)$$

In this action the spin-connection components $\omega^{(a,0b)}$ only occur linearly. They are therefore independent Lagrange multipliers leading to the geometric constraints :

$$t_{0a,}{}^a = t_0{}^{\{a,b\}} = 0: \quad \text{Carroll 4 geometry}$$

Hansen, Obers, Oling, Søgaard (2021); Henneaux, Salgado-Rebolledo (2021)
Campoleoni, Henneaux, Pekar, Pérez, Salgado-Rebolledo (2022)

Solving for the other spin-connections leads to 2d-order Carroll gravity

Electric Carroll Gravity

Henneaux (1979)

Taking the limit of $\Omega_\mu{}^{AB}$ in a first-order formulation and then pass to a second-order formulation is not the same as taking the limit directly in a second-order formulation :

$$\Omega_\mu{}^{AB}(E) \quad \rightarrow \quad \tilde{c}^2 t_0{}^{(a,b)} + \omega_\mu{}^{ab}(e)\,,\ \omega_\mu{}^{a0}(e,\tau)$$

When taking the Carroll limit this leads to electric Carroll gravity

$$S = \frac{1}{16\pi G_C} \int \mathrm{d}^4x\, e\, \left(t_0{}^{(a,b)} t_{0(a,b)} - t_{0a}{}^a t_{0b}{}^b \right)$$

Applying a Hubbard-Stratonovich transformation, the sub-leading terms are given by 2d-order magnetic Carroll gravity

Hartong (2015); Hansen, Obers, Oling, Søgaard (2021)

The electric Carroll gravity action has no first-order formulation !

Conformal Carroll Gravity I

Applying a generalized Hubbard-Stratonovich transformation one may obtain two different electric Carroll gravity theories:

1. Electric Carroll Gravity (ECG) is invariant under Carroll transformations:

$$S_{\text{ECG}} = \frac{1}{16\pi G_C} \int \mathrm{d}^4x t_{0a,}{}^{a} t_{0b,}{}^{b}$$

2. Conformal Carroll Gravity (CCG) is invariant under conformal Carroll transformations:

$$S_{\text{CCG}} = \frac{1}{16\pi G_C} \int \mathrm{d}^4x \, e \, t_0{}^{\{a,b\}} t_{0\{a,b\}}$$

Outline

Relativistic Conformal Gravity

The relativistic conformal algebra consists of translations P_A, Lorentz transformations M_{AB} plus additional dilatations D and special conformal transformations K_A

Imposing the conformal curvature constraints $\mathcal{R}_{\mu\nu}{}^A(P) = E^\nu{}_B \mathcal{R}_{\mu\nu}{}^{AB}(M) = 0$ allows one to solve for $\Omega_\mu{}^{AB}$ and the special conformal gauge field $f_\mu{}^A$ while the dilatation gauge field b_μ transforms with a shift under K_A $\rightarrow$ b_A cancels out!

We wish to couple a compensating scalar field Φ to conformal gravity as a first step towards constructing general matter couplings :

$$\delta\Phi = -\Lambda_D\Phi \quad \rightarrow \quad D_A\Phi \equiv (\partial_A + b_A)\Phi \quad \rightarrow$$

$$\delta D_A\Phi = -2\Lambda_D D_A\Phi + \Lambda_A{}^B D_B\Phi + \Lambda_{KA}\Phi \quad \rightarrow$$

$$D^A D_A\Phi \equiv \partial^A D_A\Phi + 2b^A D_A\phi - \Omega^A{}_{,A}{}^B D_B\Phi - f^A{}_A\Phi$$

$$\phi\,\partial^A\partial_A\phi \quad \Leftrightarrow \quad \Phi D^A D_A\Phi \sim f^A{}_A\Phi^2 \quad \overset{\Phi=1}{\Rightarrow} \text{ or } \overset{E_\mu{}^A \to \Phi E_\mu{}^A}{\Leftarrow} \quad R(M)$$

matter $\Leftrightarrow$ matter + geometry $\Leftrightarrow$ geometry

Conformal Carroll Gravity II

P. Concha, E. Rodríguez and O. Fierro, work in progress; see also Lovrekovic (2022)
cp. to Baiguera, Oling, Sybesma, Søgaard 2022)

The conformal Carroll algebra consists of time/space translations H/P_a, spatial rotations/Carroll boost transformations J_{ab}/G_{0a} plus additional dilatations D and vector/singlet special conformal transformations K_a/K

After imposing the curvature constraints

$$\mathcal{R}_{\mu\nu}(H) = t^c_{\mu\nu}{}^a(P) = \mathcal{R}_{\mu b}{}^{ab}(J) = \mathcal{R}_{\mu a}{}^{0a}(G) = 0$$

except for $t^{0\{a,b\}} \neq 0$ we find that

The gauge fields $(\tau_\mu\,, e_\mu{}^a)$ are independent ,

The gauge fields $(\omega_\mu{}^{ab}\,, \omega_\mu{}^{0a})$ are dependent except for $\omega^{(a,0b)}$,

The gauge fields b_a are independent but can be shifted away by K_a ,

b_0 is solved for by $b_0 = t_{0a,}{}^a$; instead $\omega_{a,}{}^{a0}$ is shifted away by K ,

The gauge fields $(g_\mu{}^a\,, f_\mu)$ can both be solved for in terms of $R(J)$ and $R(G)$ but only a combination of the two gives Magnetic Carroll Gravity

Coupling CCG to an Electric Scalar

We wish to couple, for $m = 0$, an electric Carroll scalar to CCG:

$$\mathcal{L}_{\text{electric scalar}} = -\frac{1}{2}\phi\partial_t\partial_t\phi \quad \text{with} \quad \delta\phi = -\lambda_D\phi$$

We first replace

$$\partial_t\phi \quad \to \quad D_0\phi \equiv \partial_0\phi + b_0\phi$$

Now use that $\delta b_0 = \partial_0\lambda_D \quad \to \quad \delta D_0\phi = -2\lambda_D D_0\phi \quad \to$

$$D_0 D_0\phi = (\partial_0 + 2b_0)(\partial_0 + b_0)\phi$$

Now gauge-fix $\phi = 1$, use that $b_0 = t_{0a,}{}^a$ and we obtain Electric Carroll Gravity :

$$S_{\text{ECG}} \sim \int \mathrm{d}^4x\, b_0^2 = \int \mathrm{d}^4x\, t_{0a,}{}^a t_{0b,}{}^b$$

Electric Carroll Scalar $\quad \leftrightarrow \quad$ Electric Carroll Gravity

Coupling CCG to a Magnetic Scalar

We now couple, for $m = 0$, a magnetic Carroll scalar to CCG:

$$\mathcal{L}_{\text{magnetic scalar}} = \pi\partial_t\phi + \frac{1}{2}\phi\partial^a\partial_a\phi$$

we first replace $\partial_t\phi \;\rightarrow\; D_0\phi \equiv (\partial_t + b_0)\phi$ and $\partial_a\phi \;\rightarrow\; D_a\phi \equiv (\partial_a + b_a)\phi$

Now use that $\delta b_a = \partial_a\lambda_D + \lambda_{Ka} \quad \rightarrow$

$$\delta D_a\phi = \lambda_a{}^b D_b\phi + \lambda_a{}^0 D_0\phi - 2\lambda_D D_a\phi + \lambda_{Ka}\phi \quad \rightarrow$$

$$D^a D_a\phi = (\partial^a + 2b^a)D_a\phi - \omega^{a,}{}_a{}^b D_b\phi - \omega^{a,}{}_a{}^0 D_0\phi - g^a{}_a\,\phi$$

After partially differentiating in the first three terms, the non-invariance under Carroll boosts is cancelled by assigning $\delta\pi = \lambda_0{}^a D_a\phi$

The non-invariance of the fourth term under K due to $\delta\omega^{a,}{}_a{}^0 = \lambda_K$ is cancelled, after partial differentiating, by adding the term $f_0\,\phi$

Now gauge-fix $\phi = 1 \quad \rightarrow \quad$ magnetic Carroll Gravity with $t_{0a,}{}^a = t_0{}^{\{a,b\}} = 0$

Outline

Carroll Fermions

Campoleoni, Fontanella, Mele, Rosseeel + E.B. (2023)
See talk by Lea Mele

Electric Carroll Fermions

Electric limit is straightforward but electric Carroll fermions do not transform under internal Carroll boosts

Magnetic Carroll Fermions

Instead of using a first-order formulation one uses projected fermions and gives the two different projections different scaling weights such that, after taking the limit, one obtains two Carroll fermions $\psi_\pm$ that transform under internal Carroll boosts as a reducible but undecomposable transformation :

$$\psi_- \quad \rightarrow \quad \psi_+ \quad \rightarrow \quad 0$$

There is a non-minimal formulation and a minimal formulation where the fermion kinetic term contains a Γ_5 or $\Gamma_\star$ such that the Dirac operator squares to a tachyonic Klein-Gordon operator

Supersymmetry

Fontanella, Rosseel + E.B., work in progress

Warning : supersymmetry rules can contain divergences!

Electric Supersymmetry

The commutator of two electric supersymmetry rules in a Carroll Wess-Zumino multiplet gives a time translation

Bagchi, Grumiller, Nandi (2022)

Magnetic Supersymmetry ?

One can define a Carroll limit of the action plus transformation rules corresponding to a 4D hypermultiplet yielding a finite tachyonic action that is invariant under certain fermionic transformation rules

However, these fermionic transformation rules do form a supersymmetry algebra!

The solution might be that we start from the non-minimal formulation with complex fields and take a real slice afterwards

Hartong, Ploegh, Rosseel, Van den Bleeken + E.B.(2007)

The problem also reminds a bit of constructing de Sitter supergravity that requires the use of nilpotent supermultiplets

Outline

Summary and Discussion

- we took the first step in constructing general Carroll matter couplings. A natural extension is to include interactions and supersymmetry
- we pointed out a possiblle ultra-tension limit of string theory
- we discussed a puzzle with defining magnetic supersymmetry
- one may generalize to extended objects and p-brane Carroll limits
- the analogous story for Galilei Gravity is not completely obvious. What is Electric Galilei Gravity ?

for tachyons, see Batlle, Gomis, Mezincescu, Townsend (2017)